Constrained quantization and θ-angles

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Abstract

We apply a new and mathematically rigorous method for the quantization of constrained systems to two-dimensional gauge theories. In this method, which quantizes Marsden-Weinstein symplectic reduction, the inner product on the physical state space is expressed through a certain integral over the gauge group. The present paper, the first of a series, specializes to the Minkowski theory defined on a cylinder. The integral in question is then constructed in terms of the Wiener measure on a loop group. It is shown how θ-angles emerge in the new method, and the abstract theory is illustrated in detail in an example.

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1 Introduction

1.1 Classical reduction

In Dirac's theory of constrained dynamical systems [1, 2, 3] the so-called *reduced phase space* is generically obtained by a two-step reduction procedure. In summary, the two steps of the reduction of a classical constrained system are

1. Imposing the constraints $\Phi_i = 0$; this restricts the phase space of the unconstrained system S to the *constraint hypersurface* C.

2. Quotienting by *the null foliation* $\mathcal{N}_0$ of the induced symplectic form on the constraint hypersurface C.

The reduced phase space is then $S^0 = C/\mathcal{N}_0$. This often, but not always, coincides with the phase space S^{phys} of physical degrees of freedom.

Roughly speaking, the second step undoes the underdetermination of the equations of motion on C; in gauge theories with connected gauge group physically equivalent points are identified by this step. Indeed, in a gauge theory (formulated in the temporal gauge for simplicity) the constraints are given by Gauss' law, and quotienting by the null foliation amounts to collapsing each orbit of the identity component $\mathcal{G}_0$ of the (time-independent) gauge group $\mathcal{G}$ to a point; one has $S^0 = C/\mathcal{G}_0$. If $\mathcal{G}$ is not connected, one needs to include a further step in order to arrive at S^{phys}, viz. quotienting $C/\mathcal{G}_0$ by the discrete group $\pi_0(\mathcal{G}) = \mathcal{G}/\mathcal{G}_0$. Thus $S^{\text{phys}} = S^0/\pi_0(\mathcal{G}) = C/\mathcal{G}$.

More generally, consider the case that a Lie group $\mathcal{G}$ acts canonically on S (that is, the action preserves the Poisson bracket). In the absence of certain topological obstructions [4, 5] this action is then generated by functions Φ_i (chosen relative to a basis $\{T_i\}$ of the Lie algebra $\mathbf{g}$ of $\mathcal{G}$) on S, whose Poisson brackets reproduce the Lie bracket in $\mathbf{g}$; i.e., $\{\Phi_i, \Phi_j\} = C_{ij}^k \Phi_k$. Each Φ_i plays the role of a charge, and it often happens that constraints are given by $\Phi_i = 0$ for such charges. This setting, indeed, applies in the case of a gauge group [6, 3]; also see section 2.2 below. We will refer to this situation as *the group case*; the associated reduction of S is known as *Marsden-Weinstein reduction* [4, 5, 3].

1.2 Dirac's quantum reduction

In trying to find a quantum analogue of the classical reduction procedure, Dirac [1] saw that only one of the two classical steps needs to be 'quantized'. Let us restrict ourselves to the case where all constraints are first-class (this means that all Poisson brackets $\{\Phi_i, \Phi_j\}$ vanish on C); this special case is the heart of the matter, and includes gauge theories. Assume that, through some construction, a Hilbert space $\mathcal{H}$ is given as the quantization of the (unconstrained) classical phase space S. Along with $\mathcal{H}$, which serves as the quantum state space of the unconstrained system, suppose the classical constraints have been quantized into operators $\hat{\Phi}_i$ on $\mathcal{H}$.

Dirac, then, proposed that the quantization of S^0 be given by

$$\mathcal{H}_D^0 := \{|\psi\rangle \in \mathcal{H}| \;\; \hat{\Phi}_i|\psi\rangle = 0 \,\forall i\}; \tag{1.1}$$

that is, $\mathcal{H}_D^0$ is the subspace of $\mathcal{H}$ which is annihilated by the quantum constraints. It inherits the inner product from $\mathcal{H}$, so that it becomes a Hilbert space in its own right, in which physical amplitudes may be computed. Consistency of this proposal entails that each commutator $[\hat{\Phi}_i, \hat{\Phi}_j]$ must annihilate $\mathcal{H}_D$ (i.e., the quantum theory is anomaly-free).

In the group case it suffices that the $\hat{\Phi}_i$ form a representation of the Lie algebra $\mathbf{g}$. This Lie algebra representation usually corresponds to a unitary representation U of $\mathcal{G}$, in which case the space $\mathcal{H}_D^0$ may alternatively be characterized as

$$\mathcal{H}_D^0 = \{|\psi\rangle \in \mathcal{H}| \;\; U(g)|\psi\rangle = |\psi\rangle \,\forall g \in \mathcal{G}_0\}. \tag{1.2}$$

This evidently leaves open the question how, in case $\mathcal{G}$ is disconnected, the Hilbert space $\mathcal{H}^{\text{phys}}$ is to be defined.

As we see, there is no analogue of the quotienting step of classical reduction, which would, in a way, render quantum reduction a simpler procedure than its classical counterpart. The reader will now remark that the quantum BRST procedure, at least in its operator version, does mimic its classical counterpart in being a two-step procedure as well. This is not the place to point out at what cost this

is achieved [7]; the relevant point is that the first step in quantum BRST leads to problems entirely similar to the ones encountered in the Dirac approach.

Dirac's proposal has particularly dominated the literature on canonical quantum gravity and quantum cosmology, where the so-called Hamiltonian constraint implies the controversial Wheeler-DeWitt equation. The difficulties this equation leads to are by now widely known and acknowledged [8, 9], although it is not always appreciated that most of these are merely a special instance of general problems with the Dirac (and operator-BRST) approach. The main difficulties are:

- It is very rare that all quantum constraints have 0 in their discrete spectrum, with joint eigenspace. In other words, the equations $\hat{\Phi}_i|\psi\rangle = 0$ often have no solution in $\mathcal{H}$. This situation usually occurs when the group generated by the constraints is not compact.

- If one seeks solutions outside $\mathcal{H}$, one has to construct an inner product on the space of solutions afresh. While this is possible in certain cases, there is no good prescription as to which (generalized) solutions to include.

In quantum cosmology the last problem lies behind the discussion what the 'wave function of the universe' should be [10].

1.3 A new method of constrained quantization

In view of these difficulties, and also for purely mathematical reasons, alternatives to Dirac's quantization procedure (or its BRST version) have been sought. We shall here make use of one such alternative [11, 12][1], whose essential idea is to quantize the second rather than the first step of classical reduction. This new approach turns out to work even when the Dirac (or BRST) method breaks down, reducing to it in those cases where it happens to apply. Also, one has a clean definition and construction of (weak) quantum observables (see below).

In its simplest version, this idea is implemented by manipulating the inner product $\langle\,|\,\rangle$ on $\mathcal{H}$ (which by definition is positive definite) into a sesquilinear form $\langle\,|\,\rangle_{\rm phys}$

[1] Related methods will be mentioned at the end of this subsection.

which is positive semidefinite. The construction of this form is dictated by the constraints. The form $\langle\,|\,\rangle_{\rm phys}$ will have a nonempty null space

$$\mathcal{N} = \{|\psi\rangle \in \mathcal{H}|\ \ \langle\psi|\psi\rangle_{\rm phys} = 0\}; \tag{1.3}$$

the physical state space is then given by

$$\mathcal{H}^{\rm phys} = \mathcal{H}/\mathcal{N}. \tag{1.4}$$

The inner product $\langle\,|\,\rangle^{\rm phys}$ on $\mathcal{H}^{\rm phys}$ is the one inherited from $\langle\,|\,\rangle_{\rm phys}$; it is positive definite by construction. If $V : \mathcal{H} \to \mathcal{H}^{\rm phys}$ is the canonical projection, one therefore has

$$\langle\psi|\varphi\rangle_{\rm phys} = \langle V\psi|V\varphi\rangle^{\rm phys}. \tag{1.5}$$

The Hilbert space $\mathcal{H}^{\rm phys}$ is the quantization of $S^{\rm phys}$. There is no need to pass through an intermediate space $\mathcal{H}^0$ (quantizing S^0), although it often provides insight to do so.

The set of bounded *weak quantum observables* consists of those bounded operators B on $\mathcal{H}$ which are self-adjoint with respect to the manipulated inner product, i.e., which satisfy

$$\langle\psi|B|\varphi\rangle_{\rm phys} = \overline{\langle\varphi|B|\psi\rangle}_{\rm phys} \tag{1.6}$$

for all $\psi, \varphi \in \mathcal{H}$ (here the bar stands for complex conjugation). Without the subscript 'phys' this would, of course, be the condition that B be Hermitian. A weak quantum observable B maps $\mathcal{N}$ into itself, so that its 'induced' action on the quotient $\mathcal{H}^{\rm phys}$ specifies a well-defined physical observable $B^{\rm phys}$. By definition, one has

$$B^{\rm phys}V|\psi\rangle = VB|\psi\rangle, \tag{1.7}$$

and this property completely specifies $B^{\rm phys}$ as an operator on $\mathcal{H}^{\rm phys}$.

In practice $\langle\,|\,\rangle_{\rm phys}$ is often only well-defined on a certain dense subspace $\mathcal{D} \subset \mathcal{H}$; this happens precisely when the Dirac procedure breaks down. In that case the above construction of $\mathcal{H}^{\rm phys}$ undergoes only minor modifications: the null space $\mathcal{N}$ is now defined as a subspace of $\mathcal{D}$, and the quotient $\mathcal{D}/\mathcal{N}$ has to be completed in the

inner product $\langle\,|\,\rangle^{\rm phys}$ to obtain $\mathcal{H}^{\rm phys}$. With this refinement, the key mathematical problems in the Dirac or BRST approaches are avoided. All this even works if all constraints are second-class; in fact, the classification of the constraints into first- and second-class ones is unnecessary in our procedure.

In this more general case, a weak quantum observable B is a possibly unbounded operator whose domain contains $\mathcal{D}$, and which leaves $\mathcal{D}$ stable. As in the previous paragraph, when B is a weak quantum observable the induced operator $B^{\rm phys}$ on $\mathcal{H}^{\rm phys}$ is well-defined, and represents a physical quantum observable.

Let us return to the group case, supposing that the quantum constraints generate a unitary representation $U(\mathcal{G})$ on $\mathcal{H}$. The construction of the manipulated inner product for this situation is explained in detail in [11, 12], with the following result. For the moment we assume that $\mathcal{G}$ is connected. If $\mathcal{G}$ is compact, one has $\mathcal{D} = \mathcal{H}$, and

$$\langle\psi|\varphi\rangle_{\rm phys} = \int_{\mathcal{G}_0} dg\, \langle\psi|U(g)|\varphi\rangle, \tag{1.8}$$

where dg is the Haar measure; for later reference we have written $\mathcal{G}_0$ for $\mathcal{G}$ to reflect its connectedness. This equals $\langle\psi|P_{\rm id}|\varphi\rangle$, where $P_{\rm id}$ is the projector on the subspace $\mathcal{H}_{\rm id}$ of $\mathcal{H}$ which transforms trivially under $U(\mathcal{G})$. Hence $\mathcal{N} = \mathcal{H}_{\rm id}^{\perp}$, so that $\mathcal{H}^{\rm phys} \simeq \mathcal{H}_{\rm id}$. This coincides with $\mathcal{H}_D$ of the Dirac method; cf. (1.2). A crucial property of the manipulated inner product, which is immediate from the above, is

$$\langle\psi|U(h)|\varphi\rangle_{\rm phys} = \langle\psi|\varphi\rangle_{\rm phys} \tag{1.9}$$

for all $h \in \mathcal{G}$ and all $\psi, \varphi \in \mathcal{H}$. According to (1.6) and (1.9), each $U(h)$ is a weak quantum observable, which by (1.9) and (1.7) is represented by the unit operator on $\mathcal{H}^{\rm phys}$. This suffices to prove that $\mathcal{G}_0$ acts trivially in the physical space $\mathcal{H}^{\rm phys}$.

If $\mathcal{G}$ is merely locally compact, but not compact, (and here assumed unimodular for simplicity, so that the left- and right- Haar measure coincide) the expression (1.8), and its consequence (1.9) still follows, but is only defined on a suitable dense domain $\mathcal{D} \subset \mathcal{H}$. The projection $P_{\rm id}$ and the space $\mathcal{H}_{\rm id}$ no longer exist (so that the Dirac approach would break down). However, one can successfully proceed as indicated earlier.

The case were $\mathcal{G}$ is not even locally compact, e.g. if $\mathcal{G}$ is a gauge group, will be faced in the present paper (also cf. [13, 14]). It turns out that one can still make good mathematical sense of an expression of the above kind, despite the non-existence of Haar measures on infinite-dimensional groups.

The idea of group-averaging in the context of constrained quantization goes back, at least, to Teitelboim [15, 16]; it is, of course, common practice in lattice gauge theory. The constrained quantization procedure proposed in [9] also involves expressions of the type (1.8).

1.4 Discrete reduction and θ-angles

As already remarked, the case where the gauge group $\mathcal{G}$ is disconnected is exceptional in that the reduced phase space $S^0 = C/\mathcal{G}_0$ (although symplectic) does not coincide with the physical phase space $S^{\rm phys} = S^0/\pi_0(\mathcal{G})$. The passage from S to S^0 rather than $S^{\rm phys}$ can be mimicked in quantum theory by restricting the integral in (1.8) to $\mathcal{G}_0$; this leads to a Hilbert space $\mathcal{H}^0$, which is the quantization of S^0. The passage from $\mathcal{H}^0$ to $\mathcal{H}^{\rm phys}$ would then involve a step 'quantizing' the passage from S^0 to $S^{\rm phys}$. One could, of course, postulate (1.8) also for disconnected $\mathcal{G}$, but this would overlook an important option one has available at this point.

We isolate the issue at stake by looking at the classical reduction of an arbitrary symplectic manifold S^0 by a discrete group D [17]. Unlike in general reduction, there is only one step, namely the passage to the quotient S^0/D. When D acts freely on S^0 this is a manifold, but if it doesn't S^0/D is typically an orbifold (locally a manifold) [18]. More importantly, S^0/D is symplectic (away from its possible singular points), quite unlike a quotient by a connected Lie group, which would not be symplectic, being merely a Poisson space [17].

Accordingly, there are no constraints, and $C = S^0$. The absence of constraints leads to a certain freedom in the definition of the manipulated inner product $\langle\,|\,\rangle_{\rm phys}$. Namely, having quantized S^0 by a Hilbert space $\mathcal{H}^0$, carrying a unitary representation U^0 of D, as before, we could pick an arbitrary one-dimensional unitary representation U_θ of D (where θ stands for a collection of parameters labelling such

representations), and define the manipulated inner product on $\mathcal{H}$ (or on a suitable dense subspace) by

$$\langle\psi|\varphi\rangle^{\theta}_{\rm phys} = \sum_{g\in D} U_\theta(g)\langle\psi|U^0(g)|\varphi\rangle. \tag{1.10}$$

The freedom to include U_θ is due to the fact that in discrete classical reduction the "0" in $\Phi_i = 0$, which would force U_θ to be trivial, is absent.

In case that D is nonabelian, the above limitation to one-dimensional representations U_θ yields θ-parameters only if $D/[D,D]$ is nontrivial. In the framework of [11, 12] it is, in fact, entirely possible to work with general unitary representations of D (also cf. [19]). If U_θ is defined on some Hilbert space $\mathcal{H}_\theta$, one defines the manipulated inner product $\langle\,|\,\rangle_{\rm phys}$ on $\mathcal{H}\otimes\mathcal{H}_\theta$ rather than on $\mathcal{H}$ (or, if need be, on $\mathcal{D}\otimes\mathcal{H}_\theta$) by the obvious generalization of (1.10), viz. by sesquilinear extension of

$$\langle\psi\otimes v|\varphi\otimes w\rangle_{\rm phys} = \sum_{g\in D}\langle v|U_\theta(g)|w\rangle\langle\psi|U(g)|\varphi\rangle, \tag{1.11}$$

where $\psi,\varphi\in\mathcal{H}$ and $v,w\in\mathcal{H}_\theta$. The construction of $\mathcal{H}^{\rm phys}$ then proceeds as before; the null space $\mathcal{N}$ is now a subspace of $\mathcal{H}\otimes\mathcal{H}_\theta$ (or $\mathcal{D}\otimes\mathcal{H}_\theta$). This is, for example, relevant for braid group statistics.

Returning to the general group case, with $\mathcal{G}$ disconnected, we can proceed at one stroke, avoiding the intermediate space $\mathcal{H}^0$, by replacing (1.8) by

$$\langle\psi|\varphi\rangle^{\theta}_{\rm phys} = \int_{\mathcal{G}} dg\,\tilde{U}_\theta(g)\langle\psi|U(g)|\varphi\rangle, \tag{1.12}$$

where $\tilde{U}_\theta(g) = U_\theta\circ\tau_{\mathcal{G}\to\mathcal{G}/\mathcal{G}_0}$ is defined through a one-dimensional unitary representation U_θ of $\mathcal{G}/\mathcal{G}_0$; here $\tau_{\mathcal{G}\to\mathcal{G}/\mathcal{G}_0}$ is the canonical projection from $\mathcal{G}$ to $\mathcal{G}/\mathcal{G}_0$. The inner product on the physical Hilbert space of states $\mathcal{H}^{\rm phys}$ will then depend on θ, although this θ-dependence is usually undone by a unitary transformation which puts it into the physical observables.

This procedure is equivalent to the following one, which in effect breaks the process up into two separate steps. Hence one mimicks the classical reduction process, and obtains an algorithm that in practice is often easier to implement. One first constructs $\mathcal{H}^0$ using (1.8); this only uses the restriction of $U(\mathcal{G})$ to $\mathcal{G}_0$. Since $\mathcal{G}_0$ acts trivially on $\mathcal{H}^0$ by construction, the full representation $U(\mathcal{G})$ on $\mathcal{H}$ quotients

to a representation $U^0(\mathcal{G}/\mathcal{G}_0)$ on $\mathcal{H}^0$. One then puts the manipulated inner product (1.10) (with $D = \pi_0(\mathcal{G})$) on $\mathcal{H}^0$, and proceeds to construct the physical state space $\mathcal{H}^{\rm phys}$ as before.

In other treatments of 'θ' phase factors [20, 21, 22, 19], restricted to the case of a multiply connected configuration space Q, one encounters the fundamental group $\pi_1(Q)$. To relate this to $\pi_0(\mathcal{G})$, note that a multiply connected space Q may be written as $Q = \overline{Q}/D$, with $D = \pi_1(Q)$, and $\overline{Q}$ is the universal covering space of Q. By construction, $\pi_1(\overline{Q}) = e$, and if Q is connected one has the equality $\pi_1(Q) = \pi_0(D)$. Hence we choose $S^0 = T^*\overline{Q}$, on which D acts by pull-back; the reduced space is $T^*\overline{Q}/D \simeq T^*Q$ (see below). In the opposite direction, we could start from some connected and simply connected space X (the configuration space of gauge fields, which is affine, being a case in point) and reduce $S^0 = T^*X$ by the action of a discrete group D on X (pulled back to T^*X). Since $(T^*X)/D \simeq T^*(X/D)$ for discrete D, the above-mentioned approaches would look at this as the problem of quantizing the multiply connected space $Q = X/D$. By the same argument we have $\pi_0(D) = \pi_1(X/D)$.

In all other respects our way of introducing θ-angles is profoundly different from others, and one goal of this paper is to explicitly illustrate how these angles emerge in a mathematically rigorous constrained quantization method. In the context of gauge theories the two methods of explaining such angles that are best known to field theorists (cf. [23, 24] for reviews) are, so to speak, 'orthogonal' to ours. Firstly, in the (Euclidean) path-integral method, where the θ-parameter enters through instantons, one does not integrate over the gauge group: it is precisely the goal of the Faddeev-Popov gauge fixing procedure to *avoid* such an integration. In our approach, on the other hand, all effects come from this integration. Secondly, in the Hamiltonian approach one postulates that physical states transform like $U(g)|\psi\rangle = \tilde{U}_\theta^{-1}(g)|\psi\rangle$; this generalization of Dirac's condition $U(g)|\psi\rangle = |\psi\rangle$ (cf. (1.2)) is avoided here, for the same reasons that Dirac's original condition is bypassed.

1.5 Gauge theory on a circle

We will illustrate the new technique for gauge theories on a circle $\mathbb{T} \simeq \mathbb{R}/2\pi\mathbb{Z}$. These resemble topological field theories in that the physical phase space is finite-dimensional; see [25] for the abelian case and [26, 27, 28] for the compact nonabelian case. In fact, the physical configuration space of a pure Yang-Mills theory on a circle is $G/\mathrm{ad}(G)$ (that is, the space of orbits of the adjoint action of G on itself); the derivation of this result by Marsden-Weinstein reduction will be reviewed below. For connected G this space is diffeomeorphic to T/W, where T is a maximal torus in G and W is the associated Weyl group; see [27] for a rigorous derivation in the present context, and also cf. Theorem 4.44 in [29] for the isomorphism $G/\mathrm{ad}(G) \simeq T/W$. This space is singular (but note that T/W is not an orbifold in the sense of [18]), and some care is needed in the definition of the cotangent bundle $T^*(G/\mathrm{ad}(G))$; with the correct definition this is the physical phase space S^{phys}.

To quantize according to our method, we have to face the full complexity of the problem of defining the integral in (1.8) or (1.12). It turns out that the correct choice of the gauge group $\mathcal{G}$ is to include all continuous loops in G with finite energy; in analogy with the situation on flat space [34] we might call $\mathcal{G}$ the Cameron-Martin loop group. This choice, however, leads to two (apparent) difficulties.

Firstly, being infinite-dimensional, the gauge group $\mathcal{G}$ has no Haar measure. It turns out that, heuristically speaking, the would-be "Haar measure" dg on $\mathcal{G}$ combines with a Gaussian factor in the matrix element of $U(g)$ to form a well-defined measure. This combination closely resembles the way the non-existent Lebesgue measure on the space of paths in $\mathbb{R}^3$ combines with the exponential of the kinetic term in the Euclidean action to form the Wiener measure appearing in the Feynman-Kac formula (cf., e.g., [30]). Hence one obtains essentially the Wiener measure μ_W (conditioned on loops). The Wiener measure on a loop group has appeared in the literature before in various different contexts; see [31, 32, 33].

The second difficulty is, then, that $\mathcal{G}$ has measure zero w.r.t. μ_W. While this may appear paradoxical to physicists, it is simply the well-known phenomenon that paths with finite energy are too regular to be supported by the Wiener measure.

Instead of integrating over $\mathcal{G}$ in (1.8) and (1.12), we therefore integrate over the closure $\overline{\mathcal{G}}$ of $\mathcal{G}$ in a natural norm. This closure is simply the space of all continuous loops. The representation $U(\mathcal{G})$ cannot be extended to $\overline{\mathcal{G}}$, but such an extension is not needed to define the manipulated inner product.

Using the Wiener measure on $\overline{\mathcal{G}}$, the manipulated inner product can be computed explicitly, and the structure of the physical Hilbert space $\mathcal{H}^{\rm phys}$ and the action of physical observables may be derived.

In the present paper we perform this computation when the structure group G is $U(1)$. In that case the gauge group $\mathcal{G}$ of maps from $\mathbb{T}$ to $U(1) \simeq \mathbb{T}$ is disconnected, with $\pi_0(\mathcal{G}) = \pi_1(G) = \mathbb{Z}$ (the gauge transformations are labelled by their winding number). As far as π_0 is concerned this mimicks the case where space is S^3 and $G = SU(2)$ (for here $\pi_0(\mathcal{G}) = \pi_3(G) = \mathbb{Z}$).

The construction of $\mathcal{H}^{\rm phys}$ for compact semi-simple G requires special techniques and involves fascinating mathematics, which is beyond the scope of the present paper; see [35] for a detailed treatment. As expected, the physical Hilbert space comes out to be $\mathcal{H}^{\rm phys} = L^2(G/{\rm ad}(G))$ (cf. [26, 36] for other approaches to the quantization of the Minkowski version of this model, and [37, 38, 39] for the Euclidean version), but the point of the derivation lies not so much in the result as in the method.

1.6 Acknowledgement

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2 Classical reduction

In this section we perform the Marsden-Weinstein reduction of the unphysical phase space S of Yang-Mills theory on a circle $\mathbb{T}$ to the physical phase space $S^{\rm phys}$. We assume that the structure group G is a connected compact Lie group, whose Lie algebra is denoted by $\mathfrak{g}$. Without loss of generality we take the principal G-bundle over $\mathbb{T}$, which defines the classcial setting, to be trivial, i.e., $P = \mathbb{T}\times G$. We formulate

the theory in the temporal gauge $A_0 = 0$ from the start; this partial gauge fixing is entirely innocent, and allows us to regard the gauge group $\mathcal{G}$ as consisting of maps from $\mathbb{T}$ to G. The configuration space $\mathcal{A}$ consists of certain functions from $\mathbb{T}$ to $\mathfrak{g}$. The action $g : A \to gA$ of $g \in \mathcal{G}$ on $A \in \mathcal{A}$ is given by

$$gA(\alpha) = \mathrm{Ad}(g(\alpha))A(\alpha) - dg(\alpha)\, g^{-1}(\alpha), \tag{2.13}$$

where $\mathrm{Ad}(x)A = xAx^{-1}$ stands for the adjoint action of G on $\mathfrak{g}$, and $\alpha \in \mathbb{T}$.

For the basic mathematical structure of gauge theories we refer to [40]; functional-analytic aspects are covered by [41, 42, 43]. Refs. [40] and [42] also contain most other mathematical prerequisites for this chapter.

2.1 Choice of the gauge group

It is necessary to be quite precise about the nature of the spaces $\mathcal{G}$ and $\mathcal{A}$. The gauge group $\mathcal{G}$, whose choice dictates that of $\mathcal{A}$, should not be too large, in that a space containing discontinuous gauge transformations would not reflect the topology of the bundle P. On the other hand, it should not be too small, since gauge transformations and connections that are too smooth cannot be used as the basis of a quantum theory. We will choose $\mathcal{G}$ to be the largest subspace of the group of all continuous loops for which both classical reduction can be successfully carried out, *and* the unitary representation $U(\mathcal{G})$ lying at the heart of the construction of the quantum theory is well defined.

To define $\mathcal{G}$ we recall that a compact Lie group has a Riemannian structure obtained by choosing an Ad-invariant Euclidean inner product $(\ ,\ )_{\mathfrak{g}}$ on $\mathfrak{g} \simeq \mathbb{R}^n$, and translating this from $\mathfrak{g} = T_eG$ to the tangent space of other points by the group action. Hence for a curve $\gamma : [0, 2\pi] \to G$ (and in particular for a loop g) we can define the function $|\dot{g}| : [0, 2\pi] \to \mathbb{R}$. The space $H_1(\mathbb{T}, G)$ by definition consists of those $g \in C(\mathbb{T}, G)$ whose (weak) derivative $\dot{g}$ is square-integrable in that $|\dot{g}| \in L^2([0, 2\pi], \mathbb{R})$. (Here and in what follows, $L^2([0, 2\pi], \ldots)$ is defined w.r.t the Lebesgue measure $d\alpha$, as distinct from $L^2(\mathbb{T}, \ldots)$ which is defined w.r.t. the normalized measure $d\alpha/2\pi$.) In particular, the Riemannian length of $g \in H_1(\mathbb{T}, G)$

exists. It can also be shown that such a g is absolutely continuous, and that $\dot{g}$ exists almost everywhere; see [44]. Physically, one could say that $H_1(\mathbb{T}, G)$ consists of all continuous loops with finite energy.

An alternative characterization of $H_1(\mathbb{T}, G)$ is to take the defining representation $U_d(G)$ on $\mathcal{H}_d$; the space M_d of matrices on $\mathcal{H}_d$ is a normed space, so that one can define the Hilbert space $H_1(\mathbb{T}, M_n)$ as the completion of $C^\infty(\mathbb{T}, M_n)$ in the $p = 1$ Sobolev norm. Then $H_1(\mathbb{T}, G)$ is the subspace of $H_1(\mathbb{T}, M_n)$ consisting of those functions which take values in $U_d(G)$. This endows $H_1(\mathbb{T}, G)$ with the structure of a Hilbert manifold (cf. [40, 42]). The continuous inclusion $H_1(\mathbb{T}, G) \subset C(\mathbb{T}, G)$ is then a consequence of the Sobolev embedding theorem (cf. [42]), from which it also follows that $H_1(\mathbb{T}, G)$ is not contained in any $C^p(\mathbb{T}, G)$ for $p > 0$.

The gauge group is the Hilbert Lie group

$$\mathcal{G} = H_1(\mathbb{T}, G) \tag{2.14}$$

with Lie algebra[2]

$$\mathbf{g} = H_1(\mathbb{T}, \mathfrak{g}). \tag{2.15}$$

The group operations in $\mathcal{G}$ are pointwise multiplication and inverse; these are smooth with respect to the Hilbert manifold structure of $\mathcal{G}$.

For the last point see [42, App. A]. Here $H_1(\mathbb{T}, \mathfrak{g})$ is defined analogously to $H_1(\mathbb{T}, G)$; it is a Hilbert space under the $p = 1$ Sobolev inner product

$$\langle f, g \rangle_1 = \int_{\mathbb{T}} d\alpha \, \left((f(\alpha), \overline{g(\alpha)})_{\mathfrak{g}} + (\dot{f}(\alpha), \overline{\dot{g}(\alpha)})_{\mathfrak{g}} \right). \tag{2.16}$$

One has the inclusion $H_1(\mathbb{T}, \mathfrak{g}) \subset C(\mathbb{T}, \mathfrak{g})$, and the pointwise exponential map on $\mathbf{g}$ is continuous [42].

The connectivity properties of $\mathcal{G}$ are determined by the following result.

With the gauge group $\mathcal{G}$ defined as in 2.1, and the structure group G equipped with its usual topology as a Lie group, one has

$$\pi_0(\mathcal{G}) = \pi_1(G). \tag{2.17}$$

[2]Generically Hilbert spaces are over the complex numbers, unless a real vector space is explicitly indicated, as in $L^2([0, 2\pi], \mathfrak{g})$ or $H_1(\mathbb{T}, \mathfrak{g})$, which are real Hilbert spaces.

To put this in perspective, note that one usually considers the loop group $LG = C(\mathbb{T}, G)$, equipped with the topology of uniform convergence (with respect to the metric topology of G inherited from the Riemannian structure, or from $G \simeq U_d(G)$ as above). This topology coincides with the compact-open topology, so that one has

$$\pi_0(LG) = \pi_1(G) \tag{2.18}$$

by definition of π_1.

For example, if $G = U(1)$ it follows that $\pi_0(LG) = \mathbb{Z}$; the members of a given component LG_n, $n \in \mathbb{Z}$, are labelled by the winding number of the loop. More generally, $\pi_1(G)$ is isomorphic to a discrete subgroup D of the center of the universal covering group $\overline{G}$ of G (i.e. $G = \overline{G}/D$). Under this isomorphism an element $[\delta] \in \pi_1(G)$ is the equivalence class of loops in G which are homotopic to the projection (from $\overline{G}$ to G) of a path from e to δ in $\overline{G}$.

Using (2.18), we label the components LG_δ of LG by $\delta \in D$. Since the inclusion $\mathcal{G} \subset LG$ is continuous with respect to the manifold topology on $\mathcal{G}$, (2.17) will follow if each intersection $\mathcal{G}_\delta = \mathcal{G} \cap LG_\delta$ is connected in the topology of $\mathcal{G}$; we write $\mathcal{G}_0$ for $\mathcal{G}_e$. To prove this, by the reasoning in the previous paragraph it suffices to show that any two H_1-paths in $\overline{G}$ between e and δ are homotopy-equivalent in H_1, which is obvious. Hence (2.17) follows.

An explicit description of a component $\mathcal{G}_\delta$ of $\mathcal{G}$ is as follows. Using the fact that the exponential map $\overline{\mathrm{Exp}} : \mathfrak{g} \to \overline{G}$ is surjective for compact connected Lie groups [29], we can find a $X_\delta \in \mathfrak{g}$ for which $\overline{\mathrm{Exp}}(X_\delta) = \delta$. If $[x]_D$ denotes the equivalence class in $G = \overline{G}/D$ of $x \in \overline{G}$, we have

$$\mathcal{G}_\delta = \mathcal{G}_0 g_\delta, \tag{2.19}$$

where $g_\delta(\alpha) = [\overline{\mathrm{Exp}}(X_\delta \alpha/2\pi)]_D$. In other words, any element $g_{(\delta)}$ of $\mathcal{G}_\delta$ is of the form

$$g_{(\delta)}(\alpha) = [\overline{\mathrm{Exp}}(\lambda(\alpha) + X_\delta \alpha/2\pi)]_D, \tag{2.20}$$

where $\lambda \in \mathbf{g}$; in particular, $\lambda(2\pi) = \lambda(0)$.

For example, if $G = \mathbb{T}$ one has $D = \pi_0(G) = 2\pi\mathbb{Z}$; one usually labels elements of $\mathbb{T}$ by $\alpha \in [0, 2\pi)$. The Lie algebra $\mathfrak{t}$ of $\mathbb{T}$ as well as of its covering group $\overline{\mathbb{T}} = \mathbb{R}$

is identified with $\mathbb{R}$; then $\mathrm{Exp} : \mathfrak{t} \to \mathbb{T}$ is given by $\mathrm{Exp}(X) = \exp(iX)$, whereas $\overline{\mathrm{Exp}} : \mathfrak{t} \to \mathbb{R}$ is the identity map. Hence $\delta \in \mathbb{R}$ is of the form $2\pi n$; we then have, with slight abuse of notation,

$$g_n(\alpha) = e^{in\alpha}. \tag{2.21}$$

Finally, we determine the appropriate space of connections $\mathcal{A}$; our choice is the same as the one in [27]. If $g \in H_1(\mathbb{T}, G)$ then $dg\, g^{-1} \in H_0(\mathbb{T}, \mathfrak{g}) = L^2([0, 2\pi], \mathfrak{g})$. Hence we choose $\mathcal{A} = L^2([0, 2\pi], \mathfrak{g})$ (a real Hilbert space). It can be shown that the action of $\mathcal{G}$ on $\mathcal{A}$ is smooth [42, App. A]. Since $\mathcal{A}$ is a Hilbert space, the cotangent bundle is $T^*\mathcal{A} = L^2([0, 2\pi], \mathfrak{g}^*) \times L^2([0, 2\pi], \mathfrak{g})$. We write elements of S as pairs (E, A), where E and A take values in $\mathfrak{g}^*$ and $\mathfrak{g}$, respectively. The $\mathcal{G}$-action on $\mathcal{A}$ (2.13) lifts to a smooth $\mathcal{G}$-action on S given by

$$g : (E, A) \to (\mathrm{Co}(g)E, \mathrm{Ad}(g)A - dg\, g^{-1}), \tag{2.22}$$

where we have omitted the argument α, and Co stands for the co-adjoint action of G on $\mathfrak{g}^*$. Note that $dg\, g^{-1}$ is not, in general, an element of $\mathbf{g}$. The infinitesimal transformation generated by $\lambda \in \mathbf{g}$ is

$$\lambda : (E, A) \to (E + \mathrm{Co}(\lambda)E, A - D_A\lambda), \tag{2.23}$$

where $\mathrm{Co}(\lambda)$ stands for λ taken in the co-adjoint representation, and $D_A\lambda = d\lambda + [A, \lambda] = d\lambda - \mathrm{Ad}(\lambda)A$. We may identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ through the choice of an inner product on $\mathfrak{g}$; then $\mathrm{Co}(\lambda)E$ is replaced by $\mathrm{Ad}(\lambda)E = [\lambda, E]$.

2.2 Marsden-Weinstein reduction

The procedure of Marsden-Weinstein reduction is well-defined also for infinite-dimensional (strongly) symplectic manifolds [5]; see in particular [6, 3] for Marsden-Weinstein reduction in the context of gauge theories. We here take $S = T^*\mathcal{A}$, and reduce with respect of the group action (2.22). The Poisson bracket on $C^\infty(S)$ is given by

$$\{F, G\} = \int_{\mathbb{T}} d\alpha \left(\frac{\delta F}{\delta E_a(\alpha)} \frac{\delta G}{\delta A^a(\alpha)} - \frac{\delta F}{\delta A^a(\alpha)} \frac{\delta G}{\delta E_a(\alpha)} \right), \tag{2.24}$$

where $A = A^a T_a$ and $E = E_a \theta^a$ in terms of a basis $\{T_a\}$ of $\mathfrak{g}$ and its dual basis $\{\theta^a\}$ of $\mathfrak{g}^*$. For the linear functionals $F(A) = A(f) = \langle f|A\rangle$ and $G(E) = E(g) = \langle g|E\rangle$ on S, where $f, g \in \mathcal{A}$ are smearing functions, (2.24) yields

$$\{A(f), E(g)\} = -\langle f|g\rangle. \tag{2.25}$$

In particular, $\{A(1), E(1)\} = 2\pi$.

It is clear that the action (2.22) preserves this Poisson bracket, so that it is canonical. A *momentum map* J is a function from S to the dual Lie algebra $\mathbf{g}^*$, which by definition satisfies

$$\{J_\lambda, f\} = \delta_\lambda f; \tag{2.26}$$

we write J_λ for $\langle J, \lambda\rangle$, where $\lambda \in \mathbf{g}$. Here $\delta_\lambda f$ is the infinitesimal variation under (2.23), i.e.,

$$\delta_\lambda f = \int_{\mathbb{T}} d\alpha \left(\frac{\delta f}{\delta E(\alpha)} \cdot \mathrm{Co}(\lambda) E(\alpha) - \frac{\delta f}{\delta A(\alpha)} \cdot D_A \lambda(\alpha) \right). \tag{2.27}$$

Hence a possible choice, and the one one we adopt, is

$$J_\lambda(E, A) = -\langle E|D_A \lambda\rangle, \tag{2.28}$$

where the pairing is, of course, between $L^2([0, 2\pi], \mathfrak{g}^*)$ and $L^2([0, 2\pi], \mathfrak{g})$. This momentum map is infinitesimally equivariant in the sense that

$$\{J_{\lambda_1}, J_{\lambda_2}\} = -J_{[\lambda_1, \lambda_2]}. \tag{2.29}$$

The charges Φ mentioned in the Introduction are therefore minus the components of the momentum map.

An elegant way to compute the reduced space $S^{\mathrm{phys}} = J^{-1}(0)/\mathcal{G}$ was given by Rajeev [26], and was further elaborated in [27]. All results until the end of this subsection are taken from these references; we merely add the Marsden-Weinstein reduction perspective.

Define a map $W : L^2([0, 2\pi], \mathfrak{g}) \to C([0, 2\pi], G)$ by $W(A) \equiv W_A$, given by

$$W_A(\alpha) = P \,\mathrm{Exp}\left(-\int_0^\alpha d\alpha' \, A(\alpha') \right), \tag{2.30}$$

where P denotes path-ordering, so that $W_A(\alpha)$ is indeed an element of G; note that

$$W_A(0) = e, \tag{2.31}$$

so that W takes values in the subspace $C_e([0, 2\pi], G)$ of functions satisfying[3] $f(0) = e$. The path-ordered exponential is ultimately defined as a product integral; see [45], and [27] in the present context. In our context, it coincides with the solution of the differential equation[4]

$$\left(\frac{\partial}{\partial\alpha} + A\right) W_A(\alpha) = 0, \tag{2.32}$$

with initial condition (2.31). The map W does not quite map $A \in \mathcal{A}$ into the gauge group, since $W_A(2\pi)$ is not necessarily equal to $W_A(0)$.

Although elements of S are not necessarily differentiable, the constraints $J_\lambda(E, A) = 0$ for all $\lambda \in \mathbf{g}$ force E in $(E, A) \in J^{-1}(0)$ to have the form

$$E(\alpha) = \mathrm{Co}(W_A(\alpha))E, \tag{2.33}$$

where $E \in \mathfrak{g}^*$ on the right-hand side is constant. For abelian G this simply means that $E(\alpha) = E$ is independent of α. The expression (2.33) implies that $(E, A) \in J^{-1}(0)$ satisfies Gauss' law $D_A E = 0$ (and *vice versa*).

To see the effect of passing from $J^{-1}(0)$ to S^{phys} we look at the cotangent bundle T^*G, which is canonically isomorphic to $\mathfrak{g}^* \times G$ [5]. Define $\rho : J^{-1}(0) \to T^*G$ by

$$\rho(E, A) = (E(0), W_A(2\pi)); \tag{2.34}$$

here $E(0)$ coincides with the E on the right-hand side of (2.33), and $W_A(2\pi)$ is the Wilson loop. The adjoint action of G on itself lifts to the action $y : (\theta, x) \to (\mathrm{Co}(y)\theta, \mathrm{Ad}(y)x)$ on T^*G. With respect to this lifted adjoint action, the map ρ intertwines the $\mathcal{G}$-action on S with the G-action on T^*G in that $\rho \circ g = g(0) \circ \rho$, where the $\mathcal{G}$-action on the left-hand side is given by (2.22). Since ρ is onto, the

[3] In probability theory the map $A \to W_A$ is seen as the composition $I \circ \int_0$ of the primitive $\int_0 : L^2([0, 2\pi], \mathfrak{g}) \to C([0, 2\pi], \mathfrak{g})$ and Ito's map $I = P\mathrm{Exp} : C([0, 2\pi], \mathfrak{g}) \to C([0, 2\pi], G)$; cf. [31, 32, 33].

[4] Ito's map is defined in terms of a *stochastic* differential equation similar to (2.32) but deals with much more general function A, which in our case is essentially the derivative of absolutely continuous functions.

physical phase space is

$$S^{\text{phys}} = (T^*G)/\text{Ad}(G). \tag{2.35}$$

All physical observables that only depend on A are functions of the Wilson loop; such observables define a certain commutative C^*-algebra [46]. All physical observables that polynomially depend on E are expressible in terms of the invariant elements in the universal enveloping algebra of G; the simplest such element corresponds to the energy

$$h(E, A) = \frac{1}{4\pi} \int_{\mathbb{T}} E^2, \tag{2.36}$$

where the notation E^2 includes the trace (in the co-adjoint representation).

It goes without saying that for abelian G the adjoint action is trivial, so that $S^{\text{phys}} = T^*G$ in that case.

3 Quantum reduction

3.1 Quantization of the unconstrained system

We quantize the unconstrained phase space $S = T^*\mathcal{A}$ by the standard method of second quantization. Hence we complexify the real Hilbert space $\mathcal{A} = L^2([0, 2\pi], \mathfrak{g})$ to

$$\mathcal{A}_{\mathbb{C}} = L^2([0, 2\pi], \mathfrak{g}_{\mathbb{C}}), \tag{3.37}$$

and consider the Bosonic Fock space [47]

$$\mathcal{H} = \exp(\mathcal{A}_{\mathbb{C}}) = \bigoplus_{n=0}^{\infty} \otimes_S^n \mathcal{A}_{\mathbb{C}}; \tag{3.38}$$

here $\otimes_S^n \mathcal{A}_{\mathbb{C}}$ denotes the symmetrized tensor product of n copies of $\mathcal{A}_{\mathbb{C}}$.

Of special interest are the *coherent states* $|\sqrt{\exp} A\rangle$ in $\mathcal{H}$, defined for $|A\rangle \in \mathcal{A}_{\mathbb{C}}$ by the norm-convergent series [47]

$$|\sqrt{\exp} A\rangle = \sum_n (n!)^{-1/2} \otimes^n |A\rangle; \tag{3.39}$$

the notation is motivated by the property that

$$\langle \sqrt{\exp} A | \sqrt{\exp} B\rangle = e^{\langle A|B\rangle}, \tag{3.40}$$

where $\langle A|B\rangle$ stands for the inner product in $\mathcal{A}_{\mathbb{C}}$. The importance of these vectors lies partly in the fact that one can conveniently define a unitary representation of the gauge group $\mathcal{G}$ by

$$U(g)|\sqrt{\exp}\, A\rangle = e^{-\frac{1}{2}\langle dg\, g^{-1}|dg\, g^{-1}\rangle + \langle g^{-1}dg|A\rangle}\, |\sqrt{\exp}\, gA\rangle, \tag{3.41}$$

where gA is defined in (2.13). The main term $|\sqrt{\exp}\,(\mathrm{Ad}(g)A - dg\, g^{-1})\rangle$ illustrates that this is the second quantization of the action (2.13) of $\mathcal{G}$ on $\mathcal{A}$, the other terms being present in order to guarantee that U is a unitary group representation. Various unitarily equivalent versions of this representation may be found in the literature [31, 32, 50], and have been used in the present context [36]; for a three-dimensional version cf. [13]. For later use we record the matrix element

$$\langle\sqrt{\exp}\, B|U(g)|\sqrt{\exp}\, A\rangle = e^{-\frac{1}{2}\langle dg\, g^{-1}|dg\, g^{-1}\rangle + \langle g^{-1}dg|A\rangle - \langle B|dg\, g^{-1}\rangle + \langle B|\mathrm{Ad}(g)A\rangle}. \tag{3.42}$$

For $f, g \in \mathcal{A}_{\mathbb{C}}$ the usual creation- and annihilation operators $a(f)$, $a(g)^*$ satisfy the canonical commutation relations (CCR) $[a(f), a(g)^*] = \langle f|g\rangle$; note that $a(f)$ is antilinear in f, whereas (by implication) $a(g)^*$ is linear in g. The linear span is contained in the domain of these operators, and (in the Fock representation) one has

$$a(f)|\sqrt{\exp}\, A\rangle = \langle f|A\rangle\, |\sqrt{\exp}\, A\rangle. \tag{3.43}$$

The linear functions $A(f)$ and $E(g)$ in $C^\infty(S)$ (see text after (2.24)) are quantized by

$$Q(A(f)) = \tfrac{1}{2}(a(f) + a(f)^*) \tag{3.44}$$

and

$$Q(E(g)) = -i(a(g) - a(g)^*), \tag{3.45}$$

respectively. From (2.25) and the CCR we see that

$$i[Q(A(f)), Q(E(g))] = Q(\{A(f), E(g)\}), \tag{3.46}$$

as desired in quantization theory.

3.2 Intermezzo: Wiener measure on the gauge group

The subsequent construction involved in the quantisation procedure will make use of the properties of the (conditioned) Wiener measure μ_W on $\mathcal{G}$. This measure was constructed in [31, 32, 33], and, like the Wiener measure on $\mathbb{R}^n$, is closely related to Brownian motion and the heat equation. This relation is not very important for our purpose; instead, the most efficient way to define μ_W is the following method due to L. Gross [51] (also cf. [52]). For the theory of promeasures and general measure theory in infinite-dimensional spaces we refer to the reviews [52, 53, 40]; another good reference for this subsection is section 5 of [32].

Any real Hilbert space $\mathcal{K}$ has a Gaussian promeasure μ_c defined on it, which is characterized by its Fourier transform

$$\int_{\mathcal{K}} d\mu_c(\psi) e^{i\langle\varphi|\psi\rangle} = e^{-\frac{1}{2}Q(\varphi)} = e^{-\frac{1}{2}\langle\varphi|\varphi\rangle}, \tag{3.47}$$

where $Q(\varphi) = \|\varphi\|^2$ is the covariance of μ_c. With this covariance, μ_c is the canonical Gaussian measure on $\mathcal{H}$; in general, any positive quadratic form Q can be the covariance. If $\mathcal{H}$ is finite-dimensional, μ_c is actually a measure, given by

$$d\mu_c(x^1, \ldots, x^n) = dx^1 \ldots dx^n e^{-\frac{1}{2}\|x\|^2}.$$

In general, only so-called *cylindrical functions* can be integrated with respect to a promeasure. A cylindrical function f on a Hilbert space is of the form $f = F \circ p$, where F is an integrable function on a finite-dimensional subspace, and p is the orthogonal projection onto that subspace. Eq. (3.47) provides an example: here the cylindrical function is $e^{i\langle\varphi|\psi\rangle}$. A more detailed discussion may be found in [52, 53, 40].

Given a measurable map $f : M \to N$ between two measure spaces M, N the image (or push-forward) of a measure μ on M is the measure $f_*\mu$ on N, defined by $f_*\mu(E) = \mu(f^{-1}(E))$ for all measurable subsets $E \subset N$. In case that M and N are infinite-dimensional vector spaces and μ is merely a promeasure, this definition of $f_*\mu$ initially only applies to cylinder subsets E of N. It may happen that $f_*\mu$ thus defined has a countably additive extension to the Σ-algebra generated by the cylinder sets in N, so that it can be extended to a measure on N. But even in that

case, the volume of a non-cylindrical set $E \subset N$ must be computed by approximating it with cylinder subsets, *even when* $f^{-1}(E)$ *is a cylinder set in* M.

This comment applies to the case at hand. In terms of the map W (see (2.30)) and the promeasure μ_c on $\mathcal{K} = L^2([0,2\pi],\mathfrak{g})$, the image $W_*\mu_c$ is initially a promeasure on $C_e([0,2\pi],G)$, which can be extended to a measure ν. The image E of $L^2([0,2\pi],\mathfrak{g})$ in $C_e([0,2\pi],G)$ under W (which is the subspace of continuous paths with finite energy) is not a cylinder set, and its volume should be evaluated through the approximation procedure mentioned above. It then comes out that $\nu(E) = 0$, despite the fact that $\mu_c(W^{-1}(E)) = 1$.

Let $C_{e\to x}([0,2\pi],G)$ be the space of continuous paths in G which start at e and end at x; we abbreviate this as $C_{e\to x}$. For each $x \in G$ a measure μ_x on $C_{e\to x}$ is defined by the desintegration $\nu(A) = \int_G dx\, \mu_x(\Sigma \cap C_{e\to x})$, where dx is the Haar measure on G, and Σ is a measurable subset of $C_e([0,2\pi],G)$. The special case μ_e is then a measure on the space $C_e(\mathbb{T},G)$ of continuous loops in G which start (and end) at e. If we embed G in $C(\mathbb{T},G)$ as the space of constant loops, we clearly have $C(\mathbb{T},G)/C_e(\mathbb{T},G) = G$ (as groups) and $C(\mathbb{T},G) = C_e(\mathbb{T},G) \times G$ as measure spaces. This factorization finally allows us to define the Wiener measure μ_W on $C(\mathbb{T},G)$ as the product $\mu_e \times \mu_H$.

Let $\overline{\mathcal{G}} = C(\mathbb{T},G)$ be the space of all continuous loops in G; this is the completion of $\mathcal{G}$ in the supremum norm (see [51, 52] for the general theory behind such completions in measure theory). It is clear that $\mu_W(\overline{\mathcal{G}}) = 1$, whereas the comments above imply that $\mu_W(\mathcal{G}) = 0$. We summarize this discussion by

The Wiener measure μ_W *on the extended gauge group* $\overline{\mathcal{G}}$ *is a probability measure, defined as the push-forward of the canonical Guassian promeasure on the real Hilbert space* $L^2([0,2\pi],\mathfrak{g})$ *by the 'Wilson loop' map* W *in (2.30), conditioned on the space of loops. The gauge group* $\mathcal{G}$ *of loops with finite energy has volume zero w.r.t. the Wiener measure.*

An important property of μ_W is its behaviour under translations; this was first established in [34] for the original Wiener measure on $\mathbb{R}^n$, and was proved in the

present context of loop groups by [31, 33, 54]. It is

$$d\mu_W(gh) = d\mu_W(g) \exp\left(-\tfrac{1}{2}\langle dh\, h^{-1}|dh\, h^{-1}\rangle - \langle g^{-1}dg|dh\, h^{-1}\rangle\right), \tag{3.48}$$

where $g \in \overline{\mathcal{G}}$ and $h \in \mathcal{G}$ (the translation property cannot be extended to all $h \in \overline{\mathcal{G}}$).

Another important property is that the measure is invariant with respect to $g \mapsto g^{-1}$ on all $\overline{\mathcal{G}}$ [33]. These properties, as well as the definition of μ_W, are consistent with the heuristic formula

$$\text{“}\, d\mu_W(g) = N \prod_{\alpha\in\mathbb{T}} dg(\alpha) \exp\left(-\tfrac{1}{2}\langle \frac{dg}{d\alpha}g^{-1}|\frac{dg}{d\alpha}g^{-1}\rangle\right) \,\text{”}, \tag{3.49}$$

where N is an infinite normalization constant. This formula does not make mathematical sense, since the 'Haar measure' $\prod_{\alpha\in\mathbb{T}} dg(\alpha)$ on $\mathcal{G}$ or $\overline{\mathcal{G}}$ does not exist. Nonetheless, it is sometimes useful in guessing the results of certain calculations.

3.3 The manipulated inner product

We now turn to the construction of the manipulated inner product $\langle\,|\,\rangle_{\rm phys}$. As explained in subsection 1.4, we may proceed in two stages, and first perform the quantum reduction with respect to the connected component $\mathcal{G}_0$ of the identity.

In any case, we need to determine a dense domain $\mathcal{D} \subset \mathcal{H}$ on which $\langle\,|\,\rangle_{\rm phys}$ is defined; here $\mathcal{H} = \exp(\mathcal{A}_{\mathbb{C}})$. Many different choices of $\mathcal{D}$ lead to the same physical Hilbert space; a guiding principle is computational convenience. It turns out to be appropriate to choose the following domain.

The domain $\mathcal{D} \subset \exp(\mathcal{A}_{\mathbb{C}})$ consists of the finite linear span of all coherent states of the form $|\sqrt{\exp}\, A\rangle$, where $A \in \mathcal{A}_{\mathbb{C}}$.

Following the proof of Prop. 2.2 in [47], one can show that $\mathcal{D}$ is dense $\exp(\mathcal{A}_{\mathbb{C}})$. Moreover, $\mathcal{D}$ is stable under the action of $U(g)$ for any $g \in \mathcal{G}$. The advantage of this choice will become clear later on.

It so happens that the representation U defined in (3.41) cannot be extended from $\mathcal{G}$ to $\overline{\mathcal{G}}$. Nonetheless, eqs. (1.8), (3.42), and (3.49), suggest, and almost imply, that we should define the manipulated inner product on $\mathcal{D}$ by sesquilinear extension of

$$\langle\sqrt{\exp}\, B|\sqrt{\exp}\, A\rangle_{\rm phys} = \int_{\overline{\mathcal{G}}} d\mu_W(g)\, e^{\langle g^{-1}dg|A\rangle} e^{\langle B|gA\rangle}. \tag{3.50}$$

Since $g^{-1}dg$ is not necessarily in L^2, the expressions $\langle g^{-1}dg|A\rangle$ and $\langle B|gA\rangle$ should not be interpreted as inner products in L^2, but as stochastic integrals [55, $\mathcal{S}$4.5]. In the present case these stochastic integrals reduce to Stieltjes integrals (see [34] for this remark). We shall not dwell on this point, except by saying that the following manipulations are all justified in the context of this more general notion of integration.

In any case, the postulate (3.50) is jusitified by the crucial property (1.9) (now valid on $\mathcal{D}$), which follows from (3.41), (3.50) and (3.48). Like the translation formula (3.48), this property holds for all $h \in \mathcal{G}$. It is important that $\mathcal{D}$ is stable under $U(\mathcal{G})$, since otherwise the left-hand side of (1.9) would not be defined.

4 The abelian case

4.1 Small gauge transformations

We will now look at the simplest case $G = U(1)$. First, let us reduce with respect to the space $\mathcal{G}_0$ of small gauge transformations. In the abelian case the product (3.50) simplifies to

$$\langle \sqrt{\exp} B|\sqrt{\exp} A\rangle_0 = e^{\langle B|A\rangle} \int_{\overline{\mathcal{G}}_0} d\mu_W(g) e^{\langle g^{-1}dg|A-\overline{B}\rangle}. \tag{4.51}$$

Write $g = \exp(i\lambda)$, where $\lambda(2\pi) = \lambda(0)$ for $g \in \overline{\mathcal{G}}_0$. By the definition of $\mathcal{G}_0$, the set $\{g^{-1}dg|g \in \mathcal{G}_0\}$ forms the Hilbert space $P_0^\perp L^2([0,2\pi],\mathbb{R})$ (this would no longer be true in the non-abelian case). Here P_0 is the projection onto the the constant functions; we write $P_0 A = A_0 1$, where $A_0 = (2\pi)^{-1}\int_0^{2\pi} d\alpha\, A(\alpha)$ is the zeroth Fourier mode of A, and 1 is the unit function in $L^2([0,2\pi],\mathbb{R})$. The symbol $P_0^\perp$ will denote the projection orthogonal to P_0.

Using the definition of the Wiener measure as the push-forward of the Gaussian promeasure under the map W (cf. (2.30) or (2.32)), the right-hand side of (4.51) becomes

$$e^{\langle B|A\rangle} \int_{P_0^\perp L^2([0,2\pi],\mathbb{R})} d\mu_c(\lambda) e^{\langle d\lambda|A-\overline{B}\rangle},$$

where μ_c is the canonical Gaussian measure on $P_0^\perp L^2([0,2\pi],\mathbb{R})$. This step is justified because $\overline{\mathcal{G}}_0$ is a cylinder set in $\overline{\mathcal{G}}$.

The integral itself (without the prefactor) is computed from (3.47), and yields $\exp[\frac{1}{2}(A_\perp - \overline{B_\perp})^2]$, where $A_\perp = P_0^\perp A$ etc. All in all, we obtain

$$\langle \sqrt{\exp}\, B | \sqrt{\exp}\, A \rangle_0 = e^{\frac{1}{2}(A_\perp^2 + \overline{B}_\perp^2)} e^{2\pi \overline{B}_0 A_0}, \tag{4.52}$$

where $A^2 = \langle \overline{A} | A \rangle$, etc.

By definition, the induced Hilbert space $\mathcal{H}^0$ is the quotient $\mathcal{D}/\mathcal{N}$ of $\mathcal{D}$ by the null space $\mathcal{N}$ of the manipulated inner product, completed in the inherited norm. A trick allows us to realize $\mathcal{H}^0$ in a more concrete way.

Define $V : \mathcal{D} \to L^2(\mathbb{R})$ by linear extension of

$$\langle x | V | \sqrt{exp} A \rangle = (\pi)^{-1/2} e^{\frac{1}{2}A_\perp^2} \exp\left(-\frac{x^2}{2\pi} + 2xA_0 - \pi A_0^2 \right). \tag{4.53}$$

It follows from (3.40) and a Gaussian integration that

$$\langle \varphi | \psi \rangle_0 = \langle V\varphi | V\psi \rangle, \tag{4.54}$$

for all $\psi, \varphi \in L$; cf. (1.5). Here the inner product on the right-hand side is obviously the one in $L^2(\mathbb{R})$.

This property is the whole point behind introducing the map V. For it follows that the map V has the same null space $\mathcal{N}$ as the manipulated product $\langle\, |\, \rangle_0$, so that the quotient $\mathcal{D}/\mathcal{N}$ is given by the image of $\mathcal{D}$ under V. Since the image of V is dense in $L^2(\mathbb{R})$, the closure of $V\mathcal{D}$ is obviously $L^2(\mathbb{R})$. We may therefore identify this space with $\mathcal{H}^0$.

Recall the definition of a weak quantum observable; cf. (1.6) etc. Analogously to (1.7), the induced action B^0 of a weak quantum observable B on $\mathcal{H}^0$ is given by

$$B^0 V |\psi\rangle = VB|\psi\rangle. \tag{4.55}$$

In the present situation notable examples of weak quantum observables, at least with respect to the modified inner product defined by (3.50), are $Q(A(1))$, and $Q(E(f))$ for all $f \in \mathcal{A}_\mathbb{C}$; see (3.44), (3.45). The weak observability of $Q(E(f))$ is a consequence of (3.42), (3.50), and the fact that it commutes with all gauge transformations $U(g)$. In fact, a calculation similar to the one leading to (4.52)

yields

$$\langle\sqrt{\exp}\,B|Q(E(f))|\sqrt{\exp}\,A\rangle_{\rm phys} = -2\pi i\overline{f}_0(A_0-\overline{B}_0)e^{\frac{1}{2}(A_\perp^2+\overline{B}_\perp^2)}e^{2\pi\overline{B}_0A_0}. \qquad (4.56)$$

Hence $Q(E(f))^0 = 0$ for all $f \in P_0^\perp\mathcal{A}_{\mathbb{C}}$, as was to be expected on the basis of Gauss' law. Writing the energy (2.36) as a mode expansion $E = \sum_n E_nE_{-n}/4\pi$, this means that only the zero mode contributes, leading to

$$Q(h)^0 = \frac{1}{8\pi^2}\left((a(1)-a(1)^*)^2\right)^0. \qquad (4.57)$$

Furthermore, the Wilson loop $W_A(2\pi)$ (see (2.30)) is quantized by

$$Q(W_A(2\pi)) = e^{-\frac{1}{2}i(a(1)+a(1)^*)}. \qquad (4.58)$$

These operators are evidently constructed from $a(1)$ and $a(1)^*$. From (4.55) and (4.53) we obtain

$$Q(a(1))^0 = x + \pi\frac{d}{dx}. \qquad (4.59)$$

Since the induction procedure preserves the adjoint of a weak quantum observable, it follows that

$$Q(a(1)^*)^0 = x - \pi\frac{d}{dx}, \qquad (4.60)$$

Hence in terms of the usual Schrödinger position $q = x$ and momentum $p = -id/dx$ we have

$$Q(W_A(2\pi))^0 = e^{-iq}; \qquad (4.61)$$

$$Q(h)^0 = \tfrac{1}{2}p^2 \qquad (4.62)$$

from (4.58) and (4.57), respectively. These are unbounded operators on $L^2(\mathbb{R})$, initially defined on the linear span of the usual coherent states, where they are essentially self-adjoint (cf. [48, 49] for the theory of unbounded operators on Hilbert space).

4.2 Large gauge transformations

Having arrived at the intermediate Hilbert space $\mathcal{H}^0 = L^2(\mathbb{R})$, we now complete the quantum reduction by the full group $\mathcal{G}$. As explained in subsection 1.4, the discrete

group $\pi_0(\mathcal{G}) = \mathcal{G}/\mathcal{G}_0 = 2\pi\mathbb{Z}$ acts on $\mathcal{H}^0$ through a unitary representation U^0. To compute this action, we write $U(n)$ for $U(g_n)$ and note that according to (2.21) eq. (3.41) specializes to

$$U(n)|\sqrt{\exp}\, A\rangle = e^{-\pi n^2 + 2\pi n A_0}|\sqrt{\exp}\,(A - n1)\rangle. \tag{4.63}$$

From (4.53), (4.63), and (4.55) we then infer that the corresponding realization $U^0(n)$ on $L^2(\mathbb{R})$ is simply

$$\langle x|U^0(n)|\psi\rangle = \langle x + 2\pi n|\psi\rangle. \tag{4.64}$$

The one-dimensional representations of $U_\theta(\mathcal{G}/\mathcal{G}_0)$ discussed in 1.4 are here given by

$$U_\theta(n) = e^{in\theta}, \tag{4.65}$$

where $\theta \in [0, 2\pi)$ (note that the unitary dual $\hat{\mathbb{Z}}$ of $\mathbb{Z}$ is $\hat{\mathbb{Z}} = \mathbb{T}$; one could consider any $\theta \in \mathbb{R}$, and find that all θ-dependent quantities are periodic in θ with period 2π). We then apply (1.10), which, with a convenient normalization factor, now reads

$$\langle\varphi|\psi\rangle^\theta_{\text{phys}} = 2\pi \sum_{n\in\mathbb{Z}} e^{in\theta} \int_{\mathbb{R}} dx\, \langle\varphi|x\rangle\langle x + 2\pi n|\psi\rangle. \tag{4.66}$$

This is well-defined on $\mathcal{D}' = V\mathcal{D}$, which, we recall, is the linear span of all coherent states in $L^2(\mathbb{R})$. (Other domains, such as $C_c(\mathbb{R})$ or the Schwartz space $\mathcal{S}(\mathbb{R})$ would be equally suitable, and lead to the same result.)

One then repeats the procedure that led from $\mathcal{H}$ to $\mathcal{H}^0$. In the case at hand the second step of the quantum reduction procedure is closely related to the description of the Aharonov-Bohm effect in terms of induced representations [56].

4.3 Intermezzo: induced representations revisited

More generally, whenever $\mathcal{H}^0$ is of the form $L^2(\mathsf{G})$, for some locally compact group G, and $\mathcal{G}/\mathcal{G}_0$ is a closed subgroup of G which acts on $\mathcal{H}^0$ in the right-regular representation, the reduction from $\mathcal{H}^0$ to $\mathcal{H}^{\text{phys}}$ is itself a special case of the theory of induced group representations (in the sense of Mackey; cf. [57]) as reformulated by Rieffel [58]. In this more general situation one is given a closed subgroup $H \subset \mathsf{G}$

(where G and hence H are assumed to be locally compact) and a unitary representation U_χ of H in a Hilbert space $\mathcal{H}_\chi$ (here χ is some label). These data lead to a unitary representation U^χ of G on some Hilbert space $\mathcal{H}^\chi$, said to be induced by $U_\chi(H)$ [57]. As shown in [58], one can construct U^χ and $\mathcal{H}^\chi$ as follows (also cf. [11]). For simplicity we assume that G and H are unimodular, so that left- and right-Haar measures are the same; fixing a normalization, we denote the Haar measure on G and H by dx and dh, respectively. This defines $L^2(\mathsf{G}) = L^2(\mathsf{G}, dx)$. The coset G/H then has a G-invariant measure dq, which defines $L^2(Q) = L^2(Q, dq)$.

Choose a dense subset $\mathcal{D} \subset L^2(\mathsf{G})$ as $\mathcal{D} = C_c(\mathsf{G})$, and equip $L^2(\mathsf{G}) \otimes \mathcal{H}_\chi$ with the manipulated inner product, defined by sesquilinear extension of

$$\langle \psi \otimes v | \varphi \otimes w \rangle^\chi_{\rm phys} = \int_H dh \, \langle v | U_\chi(h) | w \rangle_\chi \int_{\mathsf{G}} dx \, \langle \psi | x \rangle \langle xh | \varphi \rangle, \tag{4.67}$$

where $\langle \, | \, \rangle_\chi$ is the inner product in $\mathcal{H}_\chi$. The expression (4.67) is well-defined on $\mathcal{D} \subset \otimes \mathcal{H}_\chi$. Then choose a (measurable) cross-section $s : \mathsf{G}/H \to \mathsf{G}$, and define $V^\chi_s : \mathcal{D} \otimes \mathcal{H}_\chi \to L^2(\mathsf{G}/H) \otimes \mathcal{H}_\chi$ by

$$\langle q | V^\chi_s | \psi \otimes v \rangle = \int_H dh \, U_\chi(h) | v \rangle \langle s(q) h | \psi \rangle, \tag{4.68}$$

where $q \in \mathsf{G}/H$. A simple computation, using the invariance of dh and the property

$$\int_{\mathsf{G}/H} dq \int_H dh \, f(s(q)h) = \int_{\mathsf{G}} dx \, f(x) \tag{4.69}$$

for all $f \in C_c(\mathsf{G})$, leads to

$$\langle \Psi | \Phi \rangle^\chi_{\rm phys} = \langle V^\chi_s \Psi \otimes v | V^\chi_s \Phi \rangle \tag{4.70}$$

for all $\Psi, \Phi \in \mathcal{D} \otimes \mathcal{H}_\chi$, where the inner product on the right-hand side is in $L^2(Q) \otimes \mathcal{H}_\chi$; cf. (4.54).

Therefore, by the argument that followed (4.54), the induced space $\mathcal{H}^{\rm phys}$ (which is the closure of $\mathcal{D}/\mathcal{N}$) defined by (4.67) may be identified with $L^2(Q) \otimes \mathcal{H}_\chi$. The induced representation $U^\chi(\mathsf{G})$ is then defined by the property $U^\chi(x) V^\chi_s = V^\chi_s U_L$, where $U_L(\mathsf{G})$ is the left-regular representation on $L^2(\mathsf{G})$, tensored with the identity acting on $\mathcal{H}_\chi$.

4.4 The physical Hilbert space

Comparing (4.67) with (4.66), it is clear that this general scheme applies to the case at hand: one has $\mathsf{G} = \mathbb{R}$ and $H = 2\pi\mathbb{Z}$, so that $\mathsf{G}/H = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and $U_\chi = U_\theta$ on $\mathcal{H}_\chi = \mathbb{C}$. The Haar measure on $\mathbb{Z}$ is taken to be the counting measure times 2π, and the induced measure dq on G/H is just the Haar measure on $\mathbb{T}$. It follows that

$$\mathcal{H}^{\text{phys}} = L^2(Q) \otimes \mathcal{H}_\chi = L^2(\mathbb{T}). \tag{4.71}$$

We choose $s : \mathbb{T} \to \mathbb{R}$ to be $s(\alpha) = \alpha$ for $\alpha \in [0, 2\pi)$, upon which (4.68) reads

$$\langle \alpha | V_s^\theta | \psi \rangle = 2\pi \sum_{n \in \mathbb{Z}} e^{in\theta} \langle \alpha + 2\pi n | \psi \rangle. \tag{4.72}$$

The condition for an operator B on $L^2(\mathbb{R})$ to induce a well-defined physical operator B^{phys} on $L^2(\mathbb{T})$ is (1.6), with $\langle\,|\,\rangle_{\text{phys}}$ replaced by $\langle\,|\,\rangle_{\text{phys}}^\theta$, given by (4.66). Explicitly, this condition is equivalent to

$$\int_{\mathbb{R}} dx\, \langle \varphi | x \rangle \langle x + 2\pi n | B | \psi \rangle = \int_{\mathbb{R}} dx\, \langle \varphi | B | x \rangle \langle x + 2\pi n | \psi \rangle \tag{4.73}$$

for all $\psi, \varphi \in L$. The physical observable B^{phys} is then given by

$$B^{\text{phys}} V_s^\theta = V_s^\theta B. \tag{4.74}$$

Condition (4.73) is satisifed by all differential operators with constant coefficients, such as p and p^2, but not by the position operator q. Instead, one must consider periodic functions of x with period 2π (acting on $L^2(\mathbb{R})$ as multiplication operators). The quantization of the Wilson loop (4.61) is a case in point. From (4.74) and (4.72) we then obtain

$$\langle \alpha | Q(W_A(2\pi))^{\text{phys}} | \psi \rangle = e^{-i\alpha} \langle \alpha | \psi \rangle. \tag{4.75}$$

For any power of the Schrödinger momentum p (such as the energy (4.62)) we find the formal expression

$$\langle \alpha | (p^n)^{\text{phys}} | \psi \rangle = \left(-i \frac{d}{d\alpha} \right)^n \langle \alpha | \psi \rangle. \tag{4.76}$$

This brings us to a crucial aspect of our technique, namely the fact that our method of constructing $\mathcal{H}^{\rm phys}$ automatically selects a domain of definition for unbounded weak quantum observables. This domain is the image under V (or, in the present case, V_s^θ) of the original domain (assuming the latter to be contained in the domain $\mathcal{D}$ of the manipulated inner product). In the present case the p^n were initially defined on the domain $\mathcal{D}'$ (i.e., the linear span of all coherent states in $L^2(\mathbb{R})$). One easily verifies that the closure of p^n coincides with the closure of p^n defined on the domain $C^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. It then follows e.g. from Theorem 11.2.3 in [57] that p^n is essentially self-adjoint on $\mathcal{D}'$ for all n.

The image $\mathcal{D}_\theta$ of $\mathcal{D}'$ under V_s^θ is the domain $\mathcal{D}_\theta$, consisting of the smooth functions $\psi \in C^\infty([0, 2\pi])$ for which all derivatives $\psi^{(n)}$, $n = 0, \ldots$ satisfy the twisted boundary conditions $\psi^{(n)}(2\pi) = \exp(-i\theta)\psi^{(n)}(0)$. As in Example X.1.1 in [49] one verifies that $(p^n)^{\rm phys}$ is essentially self-adjoint on $\mathcal{D}_\theta$ for all n. This particularly applies to the enery (4.62), where $n = 2$. Hence one obtains a uniquely determined observable $Q(h)^{\rm phys}$, defined as the self-adjoint closure of $\frac{1}{2}p^2$ (initially defined on $\mathcal{D}_\theta$). Its eigenfunctions ψ_n, $n \in \mathbb{Z}$, are $\langle \alpha | \psi_n \rangle = \exp(i\alpha(n - \theta/2\pi))$, with eigenvalues $E_n = \frac{1}{2}(n - \theta/2\pi)^2$. This is one way of seeing how the θ-parameter enters the physical theory.

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